Nonmathematical Introduction*)

Every day of our lives we experience changes that occur either gradually or suddenly. We often characterize these changes as quantitative or qualitative, respectively. For example, consider the following simple experiment (Figure 1). Imagine a board supported at both ends, with a load on top. If the load λ is small enough, the board will take a bent shape with a deformation depending on the magnitude of λ and on the board's material properties (such as stiffness, K). This state of the board will remain *stable* in the sense that a small variation in the load λ (or in the stiffness K) leads to a state that is only slightly perturbed. Such a variation (described by Hooke's law) would be referred to as a quantitative change. The board is deformed within its elastic regime and will return to its original shape when the perturbation in λ is removed.

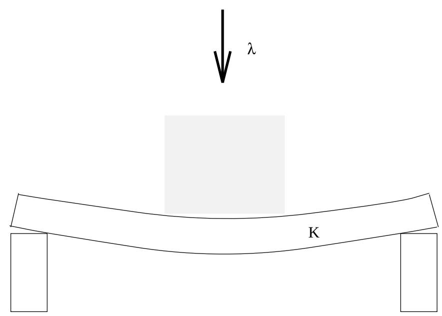


Figure 1. Bending of a board.

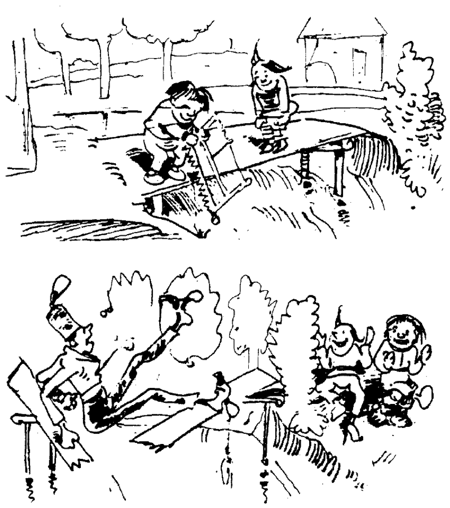
The situation changes abruptly when the load λ is increased beyond a certain *critical level* λ_0 at which the board breaks (Figure 2b). This sudden action is an example of a qualitative change; it will also take place when the material properties are changed beyond a certain limit (see Figure 2a). Suppose the shape of the board is modeled by some function (solution of an equation). Loosely speaking, we may say that there is a solution for load values $\lambda < \lambda_0$ and that this solution ceases to exist for $\lambda > \lambda_0$. The load λ and stiffness K are examples of *parameters*. The outcome of any experiment, any event, and any construction is controlled by parameters. The practical problem is to *control the state* of a system—that is, to find parameters such that the state fulfills our requirements. This role of parameters is occasionally emphasized by terms such as *control parameter*, or *design parameter*. Varying a parameter can result in a transition from a quantitative change to a qualitative change. The following pairs of verbs may serve as illustrations:

$$\begin{aligned} \text{bend} &\rightarrow \text{break} \\ \text{incline} &\rightarrow \text{tilt over} \\ \text{stretch} &\rightarrow \text{tear} \\ \text{inflate} &\rightarrow \text{burst.} \end{aligned}$$

The verbs on the left side stand for states that are stable under small *perturbations*; the response of each system is a quantitative one. This behavior ends abruptly at certain

*) after Section 1.1 from [3], Second Edition

critical values of underlying parameters. The related drastic and irreversible change is reflected by the verbs on the right side. Close to a critical threshold the system becomes most sensitive; tiny perturbations may trigger drastic changes. To control a system may mean to find parameters such that the state of the system is safe from being close to a critical threshold.



(a)

(b)

Figure 2. From W. Busch [2].

The above-mentioned problems are much too limited to cover phenomena that we will later want to denote with the term *bifurcation*. The extended range of phenomena we have in mind is indicated by the pair

$$\text{stationary state} \leftrightarrow \text{motion.}$$

Let us mention a few examples. The electric membrane potential of nerves is stationary as long as the stimulating current remains below a critical threshold; if this critical value is passed, the membrane potential begins to oscillate, resulting in nerve impulses. The motion of a semitrailer is straight for moderate speeds (assuming the rig is steered straight); if the speed exceeds a certain critical value, the vehicle tends to sway. Or take the fluttering of a flag, which will occur only if the moving air passes fast enough. Similarly, the vibration of tubes depends on the speed of the internal fluid flow and on the speed of an outer flow. This type of oscillation also occurs when obstacles, such as bridges and other high structures, are exposed to strong winds. Many other examples—too complex to be listed here—occur in combustion, fluid dynamics, and geophysics. Reference will be made to these later in the text.

The transition from a stationary state to motion, and vice versa, is also a qualitative change. Here, speaking again in terms of *solutions*—of governing equations—we have a different quality of solution on either "side" of a critical parameter. Let the parameter in question again be denoted by λ, with critical value λ_0. Thinking, for instance, in terms of wind speed, the state (e.g., of a flag or bridge) is stationary for $\lambda < \lambda_0$ and oscillatory for $\lambda > \lambda_0$. Qualitative changes may come in several steps, as indicated by the sequence stationary state

regular motion

irregular motion.

The transition from regular to irregular motion is related to the onset of turbulence, or "chaos." As a first tentative definition, we will denote a qualitative change caused by the variation of some physical (or chemical or biological, etc.) parameter λ as *branching* or *bifurcation*. We will use the same symbol λ for various kinds of parameters. Some examples of parameters are listed in the Table.

TABLE Examples of parameters.

Phenomenon	**Controlled by a typical parameter**
Bending of a rod	Load
Vibration of an engine	Frequency or imbalance
Combustion	Temperature
Voltage Collapse	Power demand
Nerve impulse	Generating potential
Superheating	Strength of external magnetic field
Oscillation of an airfoil	Speed of plane relative to air
Climatic changes	Solar radiation

Some important features that may change at bifurcations have already been mentioned. The following list summarizes various kinds of qualitative changes:

$$\begin{aligned} \text{stable} &\leftrightarrow \text{unstable} \\ \text{symmetric} &\leftrightarrow \text{asymmetric} \\ \text{stationary} &\leftrightarrow \text{periodic (regular) motion} \\ \text{regular} &\leftrightarrow \text{irregular} \\ \text{order} &\leftrightarrow \text{chaos.} \end{aligned}$$

Several of these changes may take place simultaneously in complicated ways.

The quality of solutions or states is also distinguished by their geometrical shape—that is, by their *pattern*. For example, the four patterns in Figure 3 characterize four possibilities of how the velocity of a specific combustion front varies with time (redrawn after [1]). The solution profile of Figure 3a is "flat" or stationary; this pattern stands for a uniformly propagating reaction front. Figure 3b shows a wavy pattern, representing a regularly pulsating velocity of the combustion front. The pattern of Figure 3c is again wavy but less regular, and the pattern of Figure 3d appears to be irregular (chaotic). The four different patterns of Figure 3 arise for different values of a parameter λ; new patterns form when the parameter passes critical values. This example illustrates why such bifurcation phenomena are also called *pattern formation*.

Patterns such as those depicted in Figure 3 are ubiquitous. For instance, cardiac rhythm or arrhythm is described by similar patterns. One of the possible patterns may

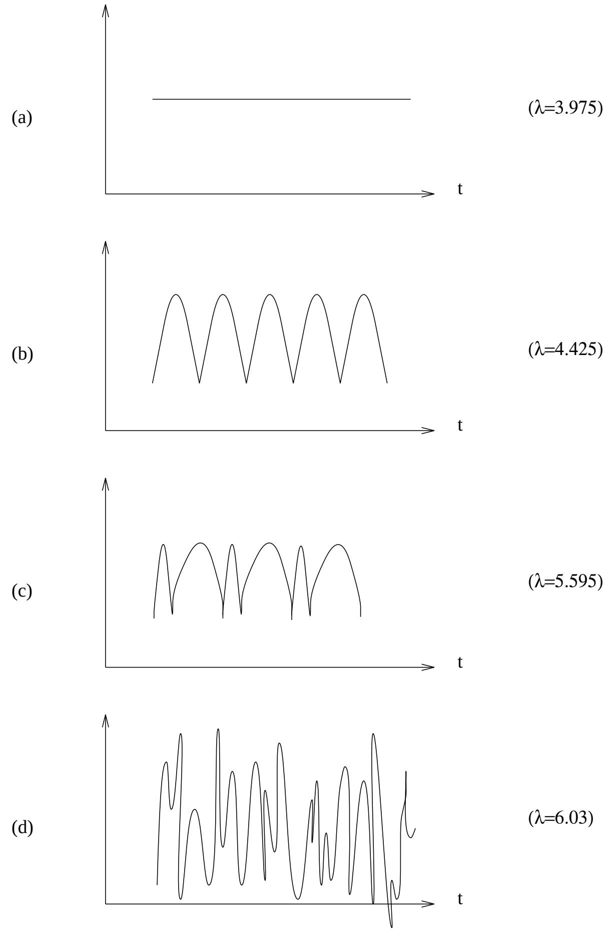


Figure 3. Velocity of a combustion front.

be more desirable than others. Hence, one faces the problem of how to *switch* patterns. By means of a proper external stimulus one can try to give the system a "kick" such that it hopefully changes its pattern to a more favorable state. For example, heart beat can be influenced by electrical stimuli. The difficulties are to decide how small a stimulus to choose and to set the best time instant for stimulation. One pattern may be more robust and harder to disturb than another pattern that may be highly sensitive and easy to excite. Before manipulating the transition among patterns, mechanisms of *pattern selection* must be studied. Which *structure* is most *attractive*? Which states are stable? For which values of the parameters is the system most sensitive?

A discussion of branching phenomena requires the language of mathematics. Tutorial 2 will review some important mathematical tools and concepts.

References

[1] Bayliss, A., Matkowsky, B.J.: Two routes to chaos in condensed phase combustion. SIAM J. Appl. Math. **50** (1990) 437–459
[2] Busch, W.: Max und Moritz. (1865) Facsimile W. Busch Gesellschaft, Hannover 1962
[3] Seydel, R.: Practical Bifurcation and Stability Analysis. Second Edition. Springer Interdisciplinary Applied Mathematics 1994 (Third Edition 2010)